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Group manifold approach to field quantisation†

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Abstract. We generalise a previously introduced group manifold approach to quantisation in order to apply it to the quantisation of free fields. The procedure is based on the consideration of infinite-dimensional groups, for which the appropriate generalisations of certain concepts of ordinary Lie group theory are introduced. The cases of the Klein-Gordon and the Proca fields are treated in detail, the latter to illustrate the treatment of constraints. The zero-mass limit of the vector fields is also briefly discussed in connection with the Stückelberg formalism.

1. Introduction

As a rule, the formalisms of geometrical quantisation [1, 2]¶ face a difficulty when they are applied to the simplest *relativistic* system, the free particle. This is because the basic quantum relations $[q^i, p_j] = \mathrm{i}\hbar\delta^i_j$ do not permit the q^i to be associated with generators of the Poincaré group as in the case of the 'non-relativistic' Galilei group (see, e.g., [3] and references therein). This difficulty is also present in the group manifold approach to geometric quantisation [4, 5], where it manifests itself as a consequence of the trivial symplectic cohomology of the Poincaré group $\mathscr{P}$, which is the starting point for the theory. Indeed, $\mathscr{P}$ does not allow for the above commutation relations or, in other words, does not admit the U(1) central extensions which are essential in the quantisation process (see [6] for a detailed discussion). This problem of the geometric 'first quantisation' of the free relativistic particle on a group manifold may be overcome by means of pseudoextensions of the Poincaré group [6] or by substituting a contraction of the conformal group with non-trivial cohomology (leading to an off-shell relativistic dynamics as an intermediate step) for the Poincaré group [3, 7]. Also, and in the broader context of (pseudo)classical systems including fermions [8], we have used [9] the $N = 2$ super-Poincaré group, which does admit central charges [10], to first quantise the Fayet-Sohnius basic matter hypermultiplet [11].

The quantisation of a free 'classical' field (the 'second' quantisation) corresponds to the quantisation of a system with infinite (continuous) degrees of freedom. The starting point for its quantisation will accordingly be an *infinite*-dimensional Lie group manifold. We may thus use this fact to avoid the non-extension theorem for $\mathscr{P}$ to

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¶ The case of fields is discussed in [2a].

quantise *relativistic* fields; this is the purpose of the present paper. As we shall see, $\mathscr{P}$ will be contained in these infinite-dimensional groups but none of its parameters will play the role of basic coordinate or conjugate momentum. (Although we shall not discuss it here, the possibility exists of considering infinite-dimensional graded groups to simultaneously incorporate bosons and fermions into the scheme.)

The paper is organised as follows. In § 2 we succinctly summarise the group quantisation formalism and apply it to the harmonic oscillator in Bargmann-Fock-Segal coordinates. In § 3 we extend the formalism to the infinite-dimensional case and use it to quantise the free Klein-Gordon (KG) field. Section 4 is devoted to the Proca field and to analysing the incorporation of constraints into the scheme. Finally, the Stückelberg formalism and the zero-mass limit are briefly discussed.

The necessary generalisations of some familiar concepts in Lie group theory to the infinite-dimensional case are performed, when required, in the main text. They are completed with three appendices on cohomology, invariant vector fields and symplectic structure.

2. The group manifold approach to quantisation and the harmonic oscillator

The group approach to quantisation (GAQ) [4, 5, 12] is a canonical algorithm which derives quantum manifolds [1] from a class of Lie groups, much in the same way that the Kirillov-Kostant method [13] obtains symplectic manifolds from Lie groups. To be more precise, the GAQ allows us to derive generalised quantum manifolds since the evolution parameter (time) is naturally included in the formalism. The necessary requirement for a group to be a quantum group $\tilde{G}$ is to have a principal bundle structure with structure group U(1). Such a condition singularises a component of the (Lie algebra $\tilde{\mathscr{G}}$-valued) left-invariant canonical 1-form θ^{L} on $\tilde{G}$ [14], the vertical component $\theta^{L_v} \equiv \Theta$, which defines a left-invariant connection-form ([14], p 103)†.

The above generalised quantum manifolds do not have to be contact manifolds. This allows for the presence on the base manifold of variables which, like time, are not included in the (symplectic) pairs of canonically conjugated variables. The presence of time allows us, unlike in the conventional formalism [1], to define generalised or full polarisation conditions which include a condition on the wavefunction which is none other than the wave equation. The variables which do not belong to the set of pairs of conjugate variables determine the vector fields which generate the *characteristic module* $\mathscr{C}_\Theta = \ker \Theta \cap \ker \mathrm{d}\Theta$; it is the quotient manifold $Q = \tilde{G}/\mathscr{C}_\Theta$ which is an ordinary quantum manifold in Souriau's sense [1] with the contact form given by $\Theta|_Q$ [4, 12]. The full polarisation, which generalises the polarisation conditions of Souriau and Kostant [1], is defined [4] as a left maximal horizontal subalgebra P containing $\mathscr{C}_\Theta$.

The Hilbert space of wavefunctions is made out of $\mathbb{C}$-valued functions on $\tilde{G}$, fulfilling the condition of being U(1)-equivariant:

$$\Xi\psi = \mathrm{i}\psi \tag{2.1a}$$

where Ξ is the vertical (U(1)) left generator, and fully polarised, i.e.

$$X^{\mathrm{P}}\psi = 0 \qquad \forall X^{\mathrm{P}} \in \mathrm{P}. \tag{2.1b}$$

† The components of θ^{L} may be easily obtained by 1-form/vector field duality considerations from the left-invariant vector fields of $\mathscr{X}^{L}(\tilde{G}) \approx T_e(\tilde{G})$ which generate the right action of $\tilde{G}$ on itself. The vertical component θ^{L_v} is dual to the U(1) vector field Ξ, $i_\Xi \theta^{L_v} = 1$, θ^{L_v} (any other left vector field) = 0.

If there is an absolute evolution parameter in $\tilde{G}$, i.e. an element in the centre of the subalgebra generated by $\mathscr{C}_\Theta$ [12], (2.1*b*) contains the evolution (Schrödinger) equation. Once the wavefunctions for the system are characterised, the basic quantum operators are defined as the right-invariant vector fields $\tilde{X}^R$†. The quantum group also provides us with an invariant measure $d\mu$ to define a scalar product for functions on the base manifold $\tilde{G}/U(1)$,

$$\langle\psi'|\psi\rangle = \int_{\tilde{G}/U(1)} \psi'^*\psi \, d\mu \tag{2.2}$$

once the U(1) parameter dependence has been factored out using (2.1*a*), on Q/U(1) or on the polarised manifold (see [12]) circumventing the delicate problem (see, e.g., [2]) of the half-forms in this way.

A particularly interesting situation appears when $\tilde{G}$ is a central extension of a group G by U(1). Then, $\tilde{G}$ has the group law

$$\tilde{g}'' = \tilde{g}'\tilde{g} = (g' * g, \zeta'\zeta \exp[i\xi(g', g)]) \tag{2.3}$$

where $\tilde{g} = (g, \zeta) \in \tilde{G}$, ζ is the U(1) parameter, $g' * g$ is the group law for G and $\xi(g', g)$ (the 2-cocycle) is an $\mathbb{R}$-valued function on $G \times G$ which characterises the central extension and in whose definition the Planck constant $\hbar$ enters‡. In this case, the central extension $\tilde{G}_{(R)}$ of G by $\mathbb{R}$ also exists, and it may be shown [4, 12] that $\tilde{G}_{(R)}$ describes the classical limit of the quantum system associated with $\tilde{G}$ [12].

Because the quantisation of a field essentially amounts to the quantisation of an infinite number of oscillators, let us first recall the group manifold quantisation of the harmonic oscillator [4, 5]. The starting point is the associated quantum group $\tilde{G}_{(m,\omega)}$ defined by the group law (henceforth we shall take $\hbar = 1$)

$$\begin{aligned}
&C'' = C' e^{-i\omega B} + R'C \qquad C, C^+ \in \mathbb{C}^3 \qquad C^+ \equiv C^* \\
&C^{+\prime\prime} = C^{+\prime} e^{i\omega B} + R'C^+ \\
&B'' = B' + B \qquad \omega, B \in \mathbb{R} \\
&R'' = R'R \qquad R \in SO(3) \\
&\zeta'' = \zeta'\zeta \exp[i\xi(g', g)] \qquad \zeta \in U(1)
\end{aligned} \tag{2.4}$$

where the 2-cocycle is given by

$$\xi(g', g) = \tfrac{1}{2}i(C'R'C^+ e^{-i\omega B} - C^{+\prime}R'C e^{i\omega B}) \tag{2.5a}$$

and the rotations are parametrised by

$$R(\varepsilon)^i_{\cdot j} = (1 - \tfrac{1}{2}\varepsilon^2)\delta^i_j + (1 - \tfrac{1}{4}\varepsilon^2)^{1/2}\eta^i_{\cdot jk}\varepsilon^k + \tfrac{1}{2}\varepsilon^i\varepsilon_j \tag{2.5b}$$

where $\eta^i_{\cdot jk}$ is the fully antisymmetric tensor. By using the familiar change of variables

$$\begin{aligned}
&C \to (m/2\omega)^{1/2}(\omega A + iV) \\
&C^+ \to (m/2\omega)^{1/2}(\omega A - iV)
\end{aligned} \tag{2.6}$$

† The action of the right-invariant vector fields of the algebra $\tilde{\mathscr{G}}$ is well defined because $[\tilde{X}^L, \tilde{X}^R] = 0$ and thus (2.1*a*, *b*) are preserved under the action of the elements of $\mathscr{X}^R(\tilde{G})$. The action of $\mathscr{X}^R(\tilde{G})$ is also well defined on the manifold $\tilde{G}/\mathscr{C}_\Theta$ because $\mathscr{C}_\Theta$ is generated by a left-invariant subalgebra.

‡ When $\tilde{G}$ has a central extension structure, the 1-form dual to the fundamental (vertical) vector field Ξ defines a horizontal subspace of the tangent space $T_{\tilde{g}}(\tilde{G})$ at each point $\tilde{g} \in \tilde{G}$. Both objects depend on the specific group law given for $\tilde{G}$, which, due to the coboundary ambiguity of the extension [6], is not unique. It is nevertheless possible to give a canonical definition for the (vertical 1-form) θ^{L_ζ} which depends only on the cohomology class and which characterises the corresponding extension cocycle [12].

it can be seen that $\tilde{G}_{(m,\omega)}$ is the central U(1) extension of the group $G_{(m,\omega)}$ which contracts to the ten-parameter Galilei group when the frequency ω of the oscillator goes to zero.

To proceed with the GAQ we first compute the left- and right-invariant vector fields (LIVF and RIVF), with the result (LIVF)

$$\tilde{X}^{L}_{C}=R\left(\frac{\partial}{\partial C}-\frac{\mathrm{i}}{2}C^{+}\Xi\right) \qquad \tilde{X}^{L}_{B}=\frac{\partial}{\partial B}-\mathrm{i}\omega C\frac{\partial}{\partial C}+\mathrm{i}\omega C^{+}\frac{\partial}{\partial C^{+}}$$
$$\tilde{X}^{L}_{C^{+}}=R\left(\frac{\partial}{\partial C^{+}}+\frac{\mathrm{i}}{2}C\Xi\right) \qquad \tilde{X}^{L}_{\varepsilon}=\varepsilon\times\frac{\partial}{\partial\varepsilon}+(1-\tfrac{1}{4}\varepsilon^{2})^{1/2}\frac{\partial}{\partial\varepsilon} \tag{2.7}$$
$$\tilde{X}^{L}_{\zeta}=\mathrm{i}\zeta\frac{\partial}{\partial\zeta}\equiv\Xi$$

and (RIVF)

$$\tilde{X}^{R}_{C}=\mathrm{e}^{-\mathrm{i}\omega B}\left(\frac{\partial}{\partial C}+\frac{\mathrm{i}}{2}C^{+}\Xi\right) \qquad X^{R}_{B}=\frac{\partial}{\partial B}$$
$$\tilde{X}^{R}_{C^{+}}=\mathrm{e}^{\mathrm{i}\omega B}\left(\frac{\partial}{\partial C^{+}}-\frac{\mathrm{i}}{2}C\Xi\right) \qquad \tilde{X}^{R}_{\zeta}=\tilde{X}^{L}_{\zeta}\equiv\Xi \tag{2.8}$$
$$\tilde{X}^{R}_{\varepsilon}=-\varepsilon\times\frac{\partial}{\partial\varepsilon}+(1-\tfrac{1}{4}\varepsilon^{2})^{1/2}\frac{\partial}{\partial\varepsilon}-C\times\frac{\partial}{\partial C}-C^{+}\times\frac{\partial}{\partial C^{+}}.$$

The commutation relations for the LIVF are

$$[\tilde{X}^{L}_{C},\tilde{X}^{L}_{C^{+}}]=\mathrm{i}\Xi \qquad [\tilde{X}^{L}_{B},\tilde{X}^{L}_{\varepsilon}]=0$$
$$[\tilde{X}^{L}_{B},\tilde{X}^{L}_{C}]=\mathrm{i}\omega\tilde{X}^{L}_{C} \qquad [\tilde{X}^{L}_{\varepsilon_i},\tilde{X}^{L}_{\varepsilon_j}]=\eta^{k}_{\cdot ij}\tilde{X}^{L}_{\varepsilon_k} \tag{2.9}$$
$$[\tilde{X}^{L}_{B},\tilde{X}^{L}_{C^{+}}]=-\mathrm{i}\omega\tilde{X}^{L}_{C^{+}} \qquad [\Xi,\text{any vector field}]=0.$$

Those for $\tilde{X}^{R}$ are the same but for a minus sign. From the duality conditions

$$\mathrm{i}_{X^{L}}\Theta=0 \qquad \forall X^{L}\neq\Xi \qquad \mathrm{i}_{\Xi}\Theta=1 \tag{2.10}$$

we obtain

$$\Theta=\tfrac{1}{2}\mathrm{i}(C^{+}\,\mathrm{d}C-C\,\mathrm{d}C^{+})-\omega CC^{+}\,\mathrm{d}B+\mathrm{d}\zeta/\mathrm{i}\zeta. \tag{2.11}$$

It is now seen that the characteristic module $\mathscr{C}_{\Theta}$ is generated by

$$\mathscr{C}_{\Theta}=\langle\tilde{X}^{L}_{B},\tilde{X}^{L}_{\varepsilon}\rangle \tag{2.12}$$

whose integral curves are the 'equations of the motion'. Indeed, the equations coming from (2.12) tell us that B is just a parameter of these trajectories, while the other variables are ε independent. The physically meaningful evolution equations are those derived from $\tilde{X}^{L}_{B}$:

$$\mathrm{d}C/\mathrm{d}s=-\mathrm{i}\omega C \qquad \mathrm{d}C^{+}/\mathrm{d}s=\mathrm{i}\omega C^{+} \qquad \mathrm{d}B/\mathrm{d}s=1 \qquad \mathrm{d}\zeta/\mathrm{d}s=0 \tag{2.13}$$

giving the trajectories

$$C(s)=C_0\,\mathrm{e}^{-\mathrm{i}\omega s} \qquad C^{+}(s)=C_0^{+}\,\mathrm{e}^{\mathrm{i}\omega s} \qquad B=s \qquad \zeta=\zeta_0. \tag{2.14}$$

Thus, the quantum manifold $Q=\tilde{G}_{(m,\omega)}/\mathscr{C}_\Theta$ is parametrised by the initial values $(\boldsymbol{C}_0, \boldsymbol{C}_0^+, \zeta_0)$ and the corresponding contact form $\Theta|_Q$ is given by

$$\Theta|_Q = \tfrac{1}{2}\mathrm{i}(\boldsymbol{C}_0^+\,\mathrm{d}\boldsymbol{C}_0 - \boldsymbol{C}_0\,\mathrm{d}\boldsymbol{C}_0^+) + \mathrm{d}\zeta_0/\mathrm{i}\zeta_0 \tag{2.15}$$

from which we obtain the symplectic (curvature) form

$$\omega \equiv \mathrm{d}(\Theta|_Q) = \mathrm{i}\,\mathrm{d}\boldsymbol{C}_0^+ \wedge \mathrm{d}\boldsymbol{C}_0. \tag{2.16}$$

We now adopt, as the polarising subalgebra P, the one including $\tilde{X}^{\mathrm{L}}_B$, $\tilde{X}^{\mathrm{L}}_{\boldsymbol{\varepsilon}}$ and $\tilde{X}^{\mathrm{L}}_{\boldsymbol{C}}$. Conditions (2.1*a*) and (2.1*b*) now imply

$$\Xi\psi = \mathrm{i}\psi \Rightarrow \psi(\boldsymbol{C}, \boldsymbol{C}^+, \boldsymbol{R}, B, \zeta) = \zeta\psi(\boldsymbol{C}, \boldsymbol{C}^+, \boldsymbol{R}, B) \tag{2.17a}$$

$$\tilde{X}^{\mathrm{L}}_{\boldsymbol{\varepsilon}}\psi = 0 \Rightarrow \psi(\boldsymbol{C}, \boldsymbol{C}^+, \boldsymbol{R}, B, \zeta) = \zeta\psi(\boldsymbol{C}, \boldsymbol{C}^+, B) \tag{2.17b}$$

$$\tilde{X}^{\mathrm{L}}_{\boldsymbol{C}}\psi = 0 \Rightarrow \psi(\boldsymbol{C}, \boldsymbol{C}^+, \boldsymbol{R}, B, \zeta) = \zeta\,\exp(-\boldsymbol{C}\boldsymbol{C}^+/2)\varphi(\boldsymbol{C}^+, B) \tag{2.17c}$$

$$\tilde{X}^{\mathrm{L}}_B\psi = 0 \Rightarrow \partial\varphi/\partial B + \mathrm{i}\omega\boldsymbol{C}^+\partial\varphi/\partial\boldsymbol{C}^+ = 0. \tag{2.17d}$$

Equation (2.17*d*) is simply the evolution equation in the Bargmann–Fock–Segal picture (see, e.g., [15]), except for the zero-point energy [4, 5]†. The natural measure turns out to induce the scalar product for complex holomorphic functions,

$$\langle\psi'|\psi\rangle = \int_{\mathbb{R}^6} \mathrm{d}\boldsymbol{C}\,\mathrm{d}\boldsymbol{C}^+\,\mathrm{e}^{-\boldsymbol{C}\boldsymbol{C}^+}\varphi'^*(\boldsymbol{C}^+, B)\varphi(\boldsymbol{C}^+, B) \tag{2.18}$$

including the weight factor $\exp(-|\boldsymbol{C}|^2)$, which naturally appears because of the polarisation condition (2.17*c*).

The quantum operators are, except for an i factor, the restriction of the RIVF to the above space of polarised wavefunctions. Because LIVF which generate $\mathscr{C}_\Theta$ commute with the RIVF the quantum operators in the '$\boldsymbol{C}_0^+$ representation' are given by

$$\begin{aligned} &\boldsymbol{a}^+ = \mathrm{i}\boldsymbol{C}_0^+ \qquad && \boldsymbol{a} = -\mathrm{i}\partial/\partial\boldsymbol{C}_0^+ \\ &E = \omega\boldsymbol{a}^+\boldsymbol{a} && \boldsymbol{L} = \mathrm{i}\boldsymbol{a}^+\times\boldsymbol{a} \end{aligned} \tag{2.19}$$

where we recognise the usual creation, destruction, energy and angular momentum operators.

3. The Klein–Gordon field

We may now discuss the simplest free relativistic field. The quantum group associated with it will have to include the Poincaré group and an infinite number of oscillators. To describe the massive real KG field we thus propose the quantum group $\tilde{G}_{\mathrm{KG}}$ defined by the following group law:

$$\begin{aligned} &\Phi''(k) = \Phi'(k)\exp(-\mathrm{i}k\Lambda'a) + \Phi(\Lambda'^{-1}k) \qquad && k\in\Omega_m^+ \\ &\Phi^{+\prime\prime}(k) = \Phi^{+\prime}(k)\exp(\mathrm{i}k\Lambda'a) + \Phi^+(\Lambda'^{-1}k) && \\ &a'' = a' + \Lambda'a && a\in \mathrm{Tr}_4 \\ &\Lambda'' = \Lambda'\Lambda && \Lambda\in\mathscr{L} \\ &\zeta'' = \zeta'\zeta\exp[\mathrm{i}\xi(g', g)] && \zeta\in U(1) \end{aligned} \tag{3.1}$$

† See [12] for a discussion of the zero-point energy. In any case, this problem is absent if we supersymmetrise the harmonic oscillator by means of a graded quantum group.

$$\xi(g', g) = \frac{i}{2}\int_{\Omega_m^+} d\Omega_k[\Phi'(k)\Phi^+(\Lambda'^{-1}k)\exp(-ik\Lambda' a)$$
$$-\Phi'^+(k)\Phi(\Lambda'^{-1}k)\exp(ik\Lambda' a)] \tag{3.2}$$
$$d\Omega_k \equiv \frac{1}{(2\pi)^3}\frac{d^3k}{2E}.$$

Because the translations a have dimensions of length, the continuous index k is clearly identified with the momentum ($\hbar = 1$). The infinite character of $\tilde{G}_{KG}$ is due to the (complex) group parameters $\Phi(k)$, $\Phi^+(k)$ labelled by $k_\mu = (E \equiv +(\boldsymbol{k}^2 + m^2)^{1/2}, \boldsymbol{k}) \in \Omega_m^+$. The action of the Poincaré subgroup on a simple redefinition of these parameters (see (3.13) below) gives the usual transformation law of the components of the KG field Fourier expansion.

It is the group parameters $\Phi(k)$, $\Phi^+(k)$ which allow us to construct the extension cocycle (3.2), which formally may be considered as a sum over an infinity of oscillator cocycles (appendix 1). It may be shown that the inequivalent 2-cocycles which may be defined on $\tilde{G}_{KG}/U(1) \equiv G_{KG}$ (i.e. the group defined by (3.1) alone; notice that G_{KG} is *not* a subgroup of $\tilde{G}_{KG}$) are all characterised by Lorentz-invariant real distributions,

$$M(k) = \alpha\delta(k^2 - \beta)\theta(\pm k^0) \qquad \alpha, \beta \in \mathbb{R} \tag{3.3}$$

much in the same way the inequivalent 2-cocycles of the Galilei group are parametrised by the mass. Equation (3.3) shows why the group parameters $\Phi(k)$, $\Phi^+(k)$ are labelled by tetramomenta on the mass shell†; in natural units, $[\Phi(k)] = [\Phi^+(k)] = (\text{mass})^{-1}$.

We now apply the GAQ as before. Because $\Phi(k)$, $\Phi^+(k)$ represent a continuum of group parameters, the vector fields will now contain functional derivatives. The LIVF and RIVF are given by (see appendix 2 and [3])

$$\tilde{X}^L_{\Phi(k)} = \delta/\delta\Phi(\Lambda k) - \tfrac{1}{2}i\Phi^+(\Lambda k)\Xi$$
$$\tilde{X}^L_{\Phi^+(k)} = \delta/\delta\Phi^+(\Lambda k) + \tfrac{1}{2}i\Phi(\Lambda k)\Xi$$
$$\tilde{X}^L_{a^\mu} = \Lambda_{\alpha\mu}\left[\frac{\partial}{\partial a_\alpha} - i\int_{\Omega_m^+} d\Omega_k\, k^\alpha\left(\Phi(k)\frac{\delta}{\delta\Phi(k)} - \Phi^+(k)\frac{\delta}{\delta\Phi^+(k)}\right)\right] \tag{3.4}$$
$$\tilde{X}^L_{\varepsilon^{\mu\nu}} \equiv \overset{L}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)\partial/\partial\varepsilon^{\alpha\beta}$$
$$\tilde{X}^L_\zeta = i\zeta\partial/\partial\zeta \equiv \Xi$$

and

$$\tilde{X}^R_{\Phi(k)} = e^{-ika}(\delta/\delta\Phi(k) + \tfrac{1}{2}i\Phi^+(k)\Xi)$$
$$\tilde{X}^R_{\Phi^+(k)} = e^{ika}(\delta/\delta\Phi^+(k) - \tfrac{1}{2}i\Phi(k)\Xi)$$
$$\tilde{X}^R_{a_\mu} = \frac{\partial}{\partial a_\mu} \tag{3.5}$$

† If k was not restricted in (3.1) to be on Ω_m^+, the characteristic module $\mathscr{C}_\Theta$ would be enlarged by the $\Phi(k)$, $\Phi^+(k)$ corresponding to $k \notin \Omega_m^+$ which would then disappear from the scheme anyway.

$$\tilde{X}^{R}_{\varepsilon^{\mu\nu}} = \overset{R}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)\frac{\partial}{\partial\varepsilon^{\alpha\beta}} + \delta^{\alpha\beta}_{\mu\nu}a_\beta\frac{\partial}{\partial a^\alpha} - \int_{\Omega^+_m} d\Omega_k\left((m_{\mu\nu}(k)\Phi(k))\frac{\delta}{\delta\Phi(k)} + (m_{\mu\nu}(k)\Phi^+(k))\frac{\delta}{\delta\Phi^+(k)}\right)$$

$$\tilde{X}^{R}_{\zeta} = \tilde{X}^{L}_{\zeta} \equiv \Xi$$

respectively, where $m_{\mu\nu}(k) \equiv \delta^{\alpha\beta}_{\mu\nu}k_\beta\partial/\partial k^\alpha|_{\Omega^+_m}$, i.e.

$$m_{0i}(k) = E\partial/\partial k^i \qquad m_{ij}(k) = k_j\partial/\partial k^i - k_i\partial/\partial k_j. \tag{3.6}$$

Their Lie brackets are given in appendix 2. We shall only give here the commutator

$$[\tilde{X}^{L}_{\Phi^+(k)}, \tilde{X}^{L}_{\Phi(k')}] = -i\Delta_{kk'}\Xi \tag{3.7}$$

where $\Delta_{kk'} \equiv (2\pi)^3 2E\delta^3(\boldsymbol{k}-\boldsymbol{k}')$, which exhibits the conjugate character of Φ and Φ^+. Notice that the functional derivative has been defined for the measure $d\Omega_k$; thus $[\delta/\delta\Phi(k)] = [\delta/\delta\Phi^+(k)] = (\text{mass})^{-1}$ as required by the dimensional homogeneity of the terms in (3.4) and (3.5).

From (3.4) and the duality conditions the generalised quantisation form Θ is derived to be

$$\Theta = \frac{i}{2}\int_{\Omega^+_m} d\Omega_k(\Phi^+(k)\,d\Phi(k) - \Phi(k)\,d\Phi^+(k)) - \int_{\Omega^+_m} d\Omega_k\, k^\mu\Phi(k)\Phi^+(k)\,da_\mu + d\zeta/i\zeta \tag{3.8}$$

by using the relations among the cotangent $T^*(\tilde{G}_{KG})$ and tangent $T(\tilde{G}_{KG})$ space coordinates

$$\frac{\delta\Phi(k)}{\delta\Phi^+(k')} = \frac{\delta\Phi^+(k)}{\delta\Phi(k')} = 0 \qquad \frac{\delta\Phi^+(k)}{\delta\Phi^+(k')} = \frac{\delta\Phi(k)}{\delta\Phi(k')} = \Delta_{kk'}$$
$$da^\mu\left(\frac{\partial}{\partial a^\nu}\right) = \delta^\mu_\nu \qquad d\varepsilon^{\alpha\beta}\left(\frac{\partial}{\partial\varepsilon^{\mu\nu}}\right) = \delta^{\alpha\beta}_{\mu\nu} \qquad d\zeta\left(\frac{\partial}{\partial\zeta'}\right) = 1. \tag{3.9}$$

From (3.8) the presymplectic form $d\Theta$ is easily derived:

$$d\Theta = i\int_{\Omega^+_m} d\Omega_k\, d\Phi^+(k)\wedge d\Phi(k) - \int_{\Omega^+_m} d\Omega_k\, k^\mu(\Phi(k)\,d\Phi^+(k) + \Phi^+(k)\,d\Phi(k))\wedge da_\mu. \tag{3.10}$$

The characteristic module is now seen to be generated by the LIVF of the Poincaré subalgebra

$$\mathscr{C}_\Theta = \langle\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{\varepsilon^{\alpha\beta}}\rangle \tag{3.11}$$

as might be expected from (3.1) and (3.2) (cf (2.4) and (2.12)). To obtain the integral manifold of the differential system (3.11) we label the respective integration parameters λ^μ and $\lambda^{\mu\nu}$. From $\tilde{X}^{L}_{\varepsilon^{\mu\nu}}$ we then get

$$\frac{d\Phi(k)}{d\lambda^{\mu\nu}} = \frac{d\Phi^+(k)}{d\lambda^{\mu\nu}} = \frac{da_\tau}{d\lambda^{\mu\nu}} = \frac{d\zeta}{d\lambda^{\mu\nu}} = 0 \qquad \frac{d\varepsilon^{\alpha\beta}}{d\lambda^{\mu\nu}} = \overset{L}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon). \tag{3.12a}$$

Similarly, $\tilde{X}^{\mathrm{L}}_{a_\mu}$ gives

$$\frac{\mathrm{d}a_\alpha}{\mathrm{d}\lambda^\mu}=\Lambda_{\alpha\mu} \qquad \frac{\mathrm{d}\Phi(k)}{\mathrm{d}\lambda^\mu}=-\mathrm{i}\Lambda_{\alpha\mu}k^\alpha\Phi(k)$$
$$\frac{\mathrm{d}\Phi^+(k)}{\mathrm{d}\lambda^\mu}=\mathrm{i}\Lambda_{\alpha\mu}k^\alpha\Phi^+(k) \qquad \frac{\mathrm{d}\varepsilon^{\alpha\beta}}{\mathrm{d}\lambda^\mu}=\frac{\mathrm{d}\zeta}{\mathrm{d}\lambda^\mu}=0. \tag{3.12b}$$

The integration gives the trajectories for the group parameters

$$\varepsilon^{\alpha\beta}=\varepsilon^{\alpha\beta}(\lambda^{\mu\nu}) \qquad \Phi(k)=\Phi_0(k)\,\mathrm{e}^{-\mathrm{i}ka}$$
$$a_\alpha=\Lambda_{\alpha\mu}\lambda^\mu \qquad \Phi^+(k)=\Phi_0^+(k)\,\mathrm{e}^{\mathrm{i}ka} \tag{3.13}$$
$$\zeta=\zeta_0$$

of which those of $\varepsilon^{\alpha\beta}$ are uninteresting because of (3.17) below.

The quantum manifold $Q=\tilde{\mathrm{G}}_{\mathrm{KG}}/\mathscr{C}_\Theta$ is parametrised solely by

$$(\Phi_0(k),\Phi_0^+(k),\zeta_0). \tag{3.14}$$

On Q, Θ and $\mathrm{d}\Theta$ adopt the form $\Theta/\mathscr{C}_\Theta\equiv\Theta|_Q$

$$\Theta|_Q=\frac{\mathrm{i}}{2}\int_{\Omega_m^+}\mathrm{d}\Omega_k(\Phi_0^+(k)\,\mathrm{d}\Phi_0(k)-\Phi_0(k)\,\mathrm{d}\Phi_0^+(k))+\mathrm{d}\zeta_0/\mathrm{i}\zeta_0 \tag{3.15a}$$

$$\mathrm{d}\Theta|_Q=\mathrm{i}\int_{\Omega_m^+}\mathrm{d}\Omega_k\,\mathrm{d}\Phi_0^+(k)\wedge\mathrm{d}\Phi_0(k) \tag{3.15b}$$

obtained by substituting (3.13) into (3.8) and (3.10) respectively. The 2-form (3.15b) defines a symplectic structure ($S=Q/\mathrm{U}(1)$, $\omega=\mathrm{d}\Theta|_S$) which together with its Poisson brackets is discussed in appendix 3.

To complete the quantisation programme we need to construct a Hilbert space $\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{KG}})$ where the Lie algebra of $\tilde{\mathrm{G}}_{\mathrm{KG}}$ is represented unitarily; the introduction of polarisation conditions will provide an irreducible representation. We take P as the subalgebra generated by (cf § 2)

$$\mathrm{P}=\langle\tilde{X}^{\mathrm{L}}_{\Phi(k)},\tilde{X}^{\mathrm{L}}_{a_\mu},\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}\rangle. \tag{3.16}$$

Thus, the elements of $\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{KG}})$ are the functionals $\psi\in\mathscr{F}[\tilde{\mathrm{G}}_{\mathrm{KG}}]$, $\psi:\tilde{\mathrm{G}}_{\mathrm{KG}}\to\mathbb{C}$, which satisfy the conditions (2.1a) and (2.1b), namely

$$\Xi\psi=\mathrm{i}\psi\Rightarrow\psi[\Phi(k),\Phi^+(k),\Lambda,a,\zeta]=\zeta\psi[\Phi(k),\Phi^+(k),\Lambda,a] \tag{3.17a}$$

$$\tilde{X}^{\mathrm{L}}_{\varepsilon^{\alpha\beta}}\psi=0\Rightarrow\psi[\Phi(k),\Phi^+(k),\Lambda,a,\zeta]=\zeta\psi[\Phi(k),\Phi^+(k),a] \tag{3.17b}$$

$$\tilde{X}^{\mathrm{L}}_{\Phi(k)}\psi=0\Rightarrow\psi=\zeta\left[\exp\left(-\tfrac{1}{2}\int_{\Omega_m^+}\mathrm{d}\Omega_k|\Phi(k)|^2\right)\right]\varphi[\Phi^+(k),a] \tag{3.17c}$$

$$\tilde{X}^{\mathrm{L}}_{a_\mu}\psi=0\Rightarrow\frac{\partial\varphi[\Phi^+(k),a]}{\partial a_\alpha}+\mathrm{i}\int_{\Omega_m^+}\mathrm{d}\Omega_{k'}\,k'^\alpha\Phi^+(k')\frac{\delta\varphi[\Phi^+(k),a]}{\delta\Phi^+(k')}=0. \tag{3.17d}$$

The last equation generalises (2.17d) to the present case. Using (3.17a–c), the general solution may be written in the form

$$\psi[\Phi(k),\Phi^+(k),\Lambda,a,\zeta]=\zeta\left[\exp\left(-\tfrac{1}{2}\int_{\Omega_m^+}\mathrm{d}\Omega_k\,\Phi(k)\Phi^+(k)\right)\right]\varphi[\Phi^+(k),a] \tag{3.18}$$

where the functional $\varphi[\Phi^+(k), a]$ is a solution of (3.17d). Its general factorised analytic form has the expression

$$\varphi[\Phi^+(k), a] = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int_{\Omega_m^+} \mathrm{d}\Omega_{k_1} \cdots \int_{\Omega_m^+} \mathrm{d}\Omega_{k_n} \varphi(k_1, \ldots, k_n) \times \Phi^+(k_1) \exp(-\mathrm{i}k_1 a) \ldots \Phi^+(k_n) \exp(-\mathrm{i}k_n a) \tag{3.19}$$

where the function $\varphi(k_1, \ldots, k_n)$ is symmetrical under the interchange of the momenta. Because $\Phi^+(k_i)\exp(-\mathrm{i}k_i a) = \Phi_0^+(k_i)$ (see (3.13)), the functional (3.19) defines the familiar functional $\varphi[\Phi_0^+]$ on S. The scalar product of two functionals ψ', ψ is given by

$$(\psi', \psi) = \int \mathrm{d}\Phi^+ \, \mathrm{d}\Phi \exp\left(-\int \mathrm{d}\Omega_k |\Phi(k)|^2\right) \varphi'^*[\Phi^+, a]\varphi[\Phi^+, a] \tag{3.20}$$

which is usually considered as the definition of the scalar product for the functionals φ', φ themselves. We recognise in (3.18) and (3.20) (cf (2.17c) and (2.18)) the Gaussian measure for the KG field which arises in a natural way in the GAQ.

The basic quantum operators are given by the RIVF (3.5). Using (3.17) we find their action on the functionals $\psi \in \mathscr{H}(\tilde{G}_{KG})$:

$$\tilde{X}^{R}_{\Phi(k)}\psi = -N\,\mathrm{e}^{-\mathrm{i}ka}\Phi^+(k)\varphi \tag{3.21a}$$

$$\tilde{X}^{R}_{\Phi^+(k)}\psi = N\,\mathrm{e}^{\mathrm{i}ka}\frac{\delta\varphi}{\delta\Phi^+(k)} \tag{3.21b}$$

$$\tilde{X}^{R}_{a_\mu}\psi = -\mathrm{i}N \int_{\Omega_m^+} \mathrm{d}\Omega_k \, k^\mu \Phi^+(k) \frac{\delta\varphi}{\delta\Phi^+(k)} \tag{3.21c}$$

$$X^{R}_{\varepsilon^{\mu\nu}}\psi = N\left(\mathrm{i}\delta^{\alpha\beta}_{\mu\nu} a_\beta \int_{\Omega_m^+} \mathrm{d}\Omega_k \, k_\alpha \Phi^+(k) \frac{\delta\varphi}{\delta\Phi^+(k)} - \int_{\Omega_m^+} \mathrm{d}\Omega_k (m_{\alpha\beta}(k)\Phi^+(k)) \frac{\delta\varphi}{\delta\Phi^+(k)}\right) \tag{3.21d}$$

where N is given by

$$N \equiv \zeta \exp\left(-\tfrac{1}{2}\int_{\Omega_m^+} \mathrm{d}\Omega_k |\Phi(k)|^2\right). \tag{3.21e}$$

The expression for the physical operators is obtained by restricting (3.21) to the functionals φ which are defined on the manifold $Q/\mathrm{U}(1)$ parametrised by $\Phi_0(k)$, $\Phi_0^+(k)$ (see (3.13)). On these functionals $\varphi = \varphi[\Phi_0^+(k)]$ (i.e. in the '$\Phi_0^+(k)$ representation') we get

$$a(\boldsymbol{k}) \equiv \mathrm{i}\tilde{X}^{R}_{\Phi^+(k)} = \mathrm{i}\delta/\delta\Phi_0^+(k) \tag{3.22a}$$

$$a^+(\boldsymbol{k}) \equiv \mathrm{i}\tilde{X}^{R}_{\Phi(k)} = -\mathrm{i}\Phi_0^+(k) \tag{3.22b}$$

$$P^\mu \equiv \mathrm{i}\tilde{X}^{R}_{a_\mu} = \int_{\Omega_m^+} \mathrm{d}\Omega_k \, a^+(\boldsymbol{k})a(\boldsymbol{k})k^\mu \tag{3.22c}$$

$$M_{\mu\nu} \equiv \mathrm{i}\tilde{X}^{R}_{\varepsilon^{\mu\nu}} = -\mathrm{i}\int_{\Omega_m^+} \mathrm{d}\Omega_k \, m_{\mu\nu}(k) a^+(\boldsymbol{k})a(\boldsymbol{k}) \tag{3.22d}$$

which correspond to the familiar destruction, creation, 4-momentum and relativistic angular momentum operators (see, e.g., [16] p 117). Because of (3.22*a*, *b*) we see that

$$[a(\boldsymbol{k}), a^+(\boldsymbol{k})] = \Delta_{\boldsymbol{k}\boldsymbol{k}'} \tag{3.23}$$

(so that a, a^+ are the customary annihilation and creation operators with relativistic normalisation) and that the operators (3.22) already arise in normal-ordered form.

Once we have defined the operators $a(\boldsymbol{k})$ and $a^+(\boldsymbol{k})$, we can construct the Fock space in the usual way by defining

$$|k_1, k_2, \ldots, k_n\rangle \equiv \frac{1}{\sqrt{n!}} a^+(\boldsymbol{k}_1)a^+(\boldsymbol{k}_2)\ldots a^+(\boldsymbol{k}_n)|0\rangle \tag{3.24a}$$

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int_{\Omega^+_{k_1}} \mathrm{d}\Omega_{k_1} \ldots \int_{\Omega^+_{k_n}} \mathrm{d}\Omega_{k_n}\, \varphi(k_1, \ldots, k_n)|k_1, \ldots, k_n\rangle \tag{3.24b}$$

where the $\psi(k_1, \ldots, k_n)$ are symmetric normalised functions:

$$\int_{\Omega^+_{k_1}} \mathrm{d}\Omega_{k_1} \int_{\Omega^+_{k_2}} \mathrm{d}\Omega_{k_2} \ldots \int_{\Omega^+_{k_n}} \mathrm{d}\Omega_{k_n} |\varphi(k_1, k_2, \ldots, k_n)|^2 = 1. \tag{3.25}$$

We thus see that the GAQ leads to both the coherent state and the Fock state representations. Both are isomorphic and we can check readily [17] the equality of the scalar products between the states $|\psi\rangle$ and their biunivocally associated functionals ψ

$$(\psi', \psi) = \langle\psi'|\psi\rangle \tag{3.26}$$

where (ψ', ψ) is the functional integral (3.20) and $\langle\psi'|\psi\rangle$ is the bra–ket scalar product

$$\langle\psi'|\psi\rangle = \sum_{n=0}^{\infty} \int_{\Omega^+_{k_1}} \mathrm{d}\Omega_{k_1} \ldots \int_{\Omega^+_{k_n}} \mathrm{d}\Omega_{k_n}\, \varphi'^*(k_1 \ldots k_n)\varphi(k_1 \ldots k_n). \tag{3.27}$$

The isomorphism (3.26) is simple in the present (free) case because the integration in (3.20) involves a Gaussian measure. In general (non-free theories) (3.26) may be considered as a definition of the functional integral in terms of the new group operator analogues (3.24*a*, *b*).

To conclude this section we mention that, although we have restricted ourselves here to the real (neutral) KG field, the complex case is completely similar. It only requires enlarging the group with the appropriate new parameters to accommodate the antiparticle sector.

4. The Proca field

The quantisation of a field whose manifestly covariant transformation properties require more components than the physical degrees of freedom adds the extra difficulty of dealing with the constraints which eliminate the unphysical components. In geometrical language, this elimination corresponds to deriving a true symplectic structure which provides the quantisation of the physical degrees of freedom, as exemplarised by the Dirac method [18].

The GAQ also provides an adequate framework to treat the class of constraints which appear when a quantum group $\tilde{G}$, itself a central extension of a group G by U(1), may be further extended, i.e. when the extension leading to $\tilde{G}$ had not exhausted the cohomology of G. In these circumstances the constraints appear as vector fields enlarging $\mathscr{C}_\Theta$ and spanning a subalgebra with non-trivial cohomology [12, 19]. We shall devote this section to illustrating these methods, using the case of the massive real vector field as an example.

With a notation similar to that of (3.1), we propose the following quantum group $\tilde{G}_P$ to describe the Proca field:

$$\begin{aligned}
\Phi''_\mu(k) &= \Phi'_\mu(k)\exp(-\mathrm{i}k\Lambda' a) + \Lambda'^{\ \nu}_{\mu\cdot}\Phi_\nu(\Lambda'^{-1}k)\\
\Phi^{+\prime\prime}_\mu(k) &= \Phi^{+\prime}_\mu(k)\exp(\mathrm{i}k\Lambda' a) + \Lambda'^{\ \nu}_{\mu\cdot}\Phi^+_\nu(\Lambda'^{-1}k)\\
a'' &= a' + \Lambda' a\\
\Lambda'' &= \Lambda'\Lambda
\end{aligned} \tag{4.1a}$$

$$\begin{aligned}
\zeta'' &= \zeta'\zeta\exp(\mathrm{i}\xi(g',g))\\
\xi(g',g) &= -\frac{\mathrm{i}}{2}\int_{\Omega^+_m} \mathrm{d}\Omega_k\, M^{\mu\sigma}(\Lambda'^{-1}k)\Lambda'^{\ \nu}_{\cdot\sigma}(\Phi^+_\mu(\Lambda'^{-1}k)\Phi'_\nu(k)\exp(-\mathrm{i}k\Lambda' a)\\
&\qquad -\Phi_\mu(\Lambda'^{-1}k)\Phi^{+\prime}_\nu(k)\exp(\mathrm{i}k\Lambda' a))\\
&\qquad M^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu/m^2
\end{aligned} \tag{4.1b}$$

where the group parameters $\Phi_\mu(k)$ and $\Phi^+_\mu(k)$ are now labelled by a vector index and three continuous ones ($k\in\Omega^+_m$). It should be noticed (see appendix 1 for a discussion of the cohomology of G_P, the group given by (4.1a) alone) that the cocycle (4.1b) can be regarded as the sum of

$$\begin{aligned}
\xi_1(g',g) = -\frac{\mathrm{i}}{2}\int_{\Omega^+_m} \mathrm{d}\Omega_k\, \Lambda'^{\nu\mu}[\Phi^+_\mu(\Lambda'^{-1}k)\Phi'_\nu(k)\exp(-\mathrm{i}k\Lambda' a)\\
-\Phi_\mu(\Lambda'^{-1}k)\Phi^{+\prime}_\nu(k)\exp(\mathrm{i}k\Lambda' a)]
\end{aligned} \tag{4.2}$$

which may be considered as the 'free' one, that is, as corresponding to a vector field with no constraints, and

$$\begin{aligned}
\xi_2(g',g) = \frac{\mathrm{i}}{2m^2}\int_{\Omega^+_m} \mathrm{d}\Omega_k (\Lambda'^{-1}k)^\mu k^\nu[\Phi^+_\mu(\Lambda'^{-1}k)\Phi'_\nu(k)\exp(-\mathrm{i}k\Lambda' a)\\
-\Phi_\mu(\Lambda'^{-1}k)\Phi^{+\prime}_\nu(k)\exp(\mathrm{i}k\Lambda' a)]
\end{aligned} \tag{4.3}$$

which, once added to (4.2), gives the transverse projector $M^{\mu\sigma}(k)$. We thus see that the 'longitudinal' part of both $\Phi_\mu(k)$ and $\Phi^+_\mu(k)$ is not involved in the extension of G_P.

We now proceed with the GAQ as in the previous section. The LIVF are given by

$$\begin{aligned}
\tilde{X}^{\mathrm{L}}_{\Phi_\alpha(k)} &= \Lambda_{\mu\cdot}^{\ \alpha}(\delta/\delta\Phi_\mu(\Lambda k) + \tfrac{1}{2}\mathrm{i}M^{\alpha\mu}(\Lambda k)\Phi^+_\sigma(\Lambda k)\Xi)\\
\tilde{X}^{\mathrm{L}}_{\Phi^+_\alpha(k)} &= \Lambda_{\mu\cdot}^{\ \alpha}(\delta/\delta\Phi^+_\mu(\Lambda k) - \tfrac{1}{2}\mathrm{i}M^{\alpha\mu}(\Lambda k)\Phi_\sigma(\Lambda k)\Xi)\\
\tilde{X}^{\mathrm{L}}_{a_\mu} &= \Lambda_{\nu\cdot}^{\ \mu}\left[\frac{\partial}{\partial a_\nu} - \mathrm{i}\int_{\Omega^+_m}\mathrm{d}\Omega_k\, k^\nu\left(\Phi_\sigma(k)\frac{\delta}{\delta\Phi_\sigma(k)} - \Phi^+_\sigma(k)\frac{\delta}{\delta\Phi^+_\sigma(k)}\right)\right]\\
\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}} &\equiv \overset{\mathrm{L}}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)\frac{\partial}{\partial\varepsilon^{\alpha\beta}}\\
\tilde{X}^{\mathrm{L}}_\zeta &= \mathrm{i}\zeta\frac{\partial}{\partial\zeta} \equiv \Xi
\end{aligned} \tag{4.4}$$

and the RIVF by

$$\tilde{X}^{R}_{\Phi_\alpha(k)} = e^{-ika}(\delta/\delta\Phi_\alpha(k) - \tfrac{1}{2}iM^{\alpha\sigma}(k)\Phi^+_\sigma(k)\Xi) \qquad \tilde{X}^{R}_{a_\mu} = \partial/\partial a_\mu$$

$$\tilde{X}^{R}_{\Phi^+_\alpha(k)} = e^{ika}(\delta/\delta\Phi^+_\alpha(k) + \tfrac{1}{2}iM^{\alpha\sigma}(k)\Phi_\sigma(k)\Xi) \qquad \tilde{X}^{R}_{\zeta} = \tilde{X}^{L}_{\zeta} = \Xi$$

$$\begin{aligned}\tilde{X}^{R}_{\varepsilon^{\mu\nu}} = {}& \overset{R}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)\frac{\partial}{\partial\varepsilon^{\alpha\beta}} + \delta^{\alpha\beta}_{\mu\nu}a_\alpha\frac{\partial}{\partial a_\beta} \\ & - \int_{\Omega^+_m} d\Omega_k\left((m_{\mu\nu}(k)\Phi_\sigma(k))\frac{\delta}{\delta\Phi_\sigma(k)} + (m_{\mu\nu}(k)\Phi^+_\sigma(k))\frac{\delta}{\delta\Phi^+_\sigma(k)}\right) \\ & + \delta^{\sigma\tau}_{\mu\nu}\int_{\Omega^+_m} d\Omega_k\left(\Phi_\tau(k)\frac{\delta}{\delta\Phi^\sigma(k)} + \Phi^+_\tau(k)\frac{\delta}{\delta\Phi^{+\sigma}(k)}\right).\end{aligned} \tag{4.5}$$

Their Lie brackets are given in appendix 2. We notice here that the commutator

$$[\tilde{X}^{L}_{\Phi^+_\mu(k)}, \tilde{X}^{L}_{\Phi_\nu(k')}] = iM^{\mu\nu}(k)\Delta_{kk'}\Xi \tag{4.6}$$

does not correspond to the commutation relations of canonically conjugated variables. $M^{\mu\nu}(k)$ is singular and indicates the presence of constraints.

The calculation of the generalised quantisation form is performed from (4.4) as before. The result is

$$\begin{aligned}\Theta = {}& \frac{i}{2}\int_{\Omega^+_m} d\Omega_k\, M^{\tau\sigma}(k)(\Phi_\tau(k)\, d\Phi^+_\sigma(k) - \Phi^+_\sigma(k)\, d\Phi_\tau(k)) \\ & - \int_{\Omega^+_m} d\Omega_k\, M^{\tau\sigma}(k)k^\mu\Phi_\tau(k)\Phi_\sigma(k)\, da_\mu + d\zeta/i\zeta\end{aligned} \tag{4.7}$$

from which we get

$$\begin{aligned}d\Theta = {}& i\int_{\Omega^+_m} d\Omega_k\, M^{\tau\sigma}(k)(d\Phi_\sigma(k)\wedge d\Phi^+_\tau(k)) \\ & - \int_{\Omega^+_m} d\Omega_k\, M^{\tau\sigma}(k)k^\mu(\Phi_\tau(k)\, d\Phi^+_\sigma(k) + \Phi^+_\tau(k)\, d\Phi_\sigma(k))\wedge da_\mu.\end{aligned} \tag{4.8}$$

Using (4.7) and (4.8) we may now calculate $\mathscr{C}_\Theta$. We find that $\mathscr{C}_\Theta$ is generated by

$$\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{\varepsilon^{\mu\nu}} \tag{4.9}$$

(i.e. the LIVF of the Poincaré subalgebra) *plus*

$$\mathscr{C}(k) \equiv \frac{k_\sigma}{m}\frac{\delta}{\delta\Phi_\sigma(k)} \qquad \mathscr{C}^+(k) \equiv \frac{k_\sigma}{m}\frac{\delta}{\delta\Phi^+_\sigma(k)}. \tag{4.10}$$

Let us now integrate $\mathscr{C}_\Theta$. Let λ^μ, $s(k)$, $s^+(k)$ be the parameters of the integral curves of $\tilde{X}^{L}_{a_\mu}$, $\mathscr{C}(k)$, $\mathscr{C}^+(k)$. We shall ignore $\tilde{X}^{L}_{\varepsilon^{\mu\nu}}$ because we have already seen that there is no contribution to Q from the Lorentz part. From $\tilde{X}^{L}_{a_\mu}$ we obtain

$$\begin{aligned}&\frac{da_\nu}{d\lambda^\mu} = \Lambda_{\nu\mu} \qquad \frac{d\Phi_\sigma(k)}{d\lambda^\mu} = -i\Lambda_{\tau\mu}k^\tau\Phi_\sigma(k) \\ &\frac{d\Phi^+_\sigma(k)}{d\lambda^\mu} = i\Lambda_{\tau\mu}k^\tau\Phi^+_\sigma(k) \qquad \frac{d\varepsilon^{\alpha\beta}}{d\lambda^\mu} = 0 \qquad \frac{d\zeta}{d\lambda^\mu} = 0\end{aligned} \tag{4.11}$$

and from $\mathscr{C}(k)$, $\mathscr{C}^+(k)$

$$\frac{\mathrm{d}\Phi_\sigma(k)}{\mathrm{d}s(k)}=\frac{k_\sigma}{m} \qquad \frac{\mathrm{d}\Phi_\sigma^+(k)}{\mathrm{d}s^+(k)}=\frac{k_\sigma}{m}$$

$$\frac{\mathrm{d}\Phi_\sigma^+(k)}{\mathrm{d}s(k)}=\frac{\mathrm{d}a_\mu}{\mathrm{d}s(k)}=\frac{\mathrm{d}\varepsilon^{\alpha\beta}}{\mathrm{d}s(k)}=\frac{\mathrm{d}\zeta}{\mathrm{d}s(k)}=0 \tag{4.12}$$

$$\frac{\mathrm{d}\Phi_\sigma(k)}{\mathrm{d}s^+(k)}=\frac{\mathrm{d}a_\mu}{\mathrm{d}s^+(k)}=\frac{\mathrm{d}\varepsilon^{\alpha\beta}}{\mathrm{d}s^+(k)}=\frac{\mathrm{d}\zeta}{\mathrm{d}s^+(k)}=0.$$

The integration of (4.11) and (4.12) is performed by taking the results of one of them as the initial conditions for the other. Beginning with (4.11), we obtain

$$\Phi_\mu(k)=h_\mu(k)\exp(-\mathrm{i}ka) \qquad \varepsilon^{\alpha\beta}=\varepsilon^{\alpha\beta}(\lambda^{\mu\nu})+f^{\alpha\beta}$$

$$\Phi_\mu^+(k)=h_\mu^+(k)\exp(\mathrm{i}ka) \qquad a_\mu=\Lambda_{\nu\mu}\lambda^\nu+f_\mu \tag{4.13}$$

$$\zeta=\zeta_0$$

where $h_\mu(k)$, $h_\mu^+(k)$, $f^{\alpha\beta}$ and f_μ could still be functions of $s(k)$ and $s^+(k)$ to be determined by (4.12). Taking $\lambda^{\mu\nu}=0$, $\lambda^\nu=0$ as initial conditions, (4.12) then gives

$$\frac{\mathrm{d}h_\mu(k)}{\mathrm{d}s(k)}=\frac{k_\mu}{m} \qquad \frac{\mathrm{d}h_\mu(k)}{\mathrm{d}s^+(k)}=0 \tag{4.14a}$$

$$\frac{\mathrm{d}h_\mu^+(k)}{\mathrm{d}s(k)}=0 \qquad \frac{\mathrm{d}h_\mu^+(k)}{\mathrm{d}s^+(k)}=\frac{k_\mu}{m} \tag{4.14b}$$

$$\frac{\mathrm{d}f^{\alpha\beta}}{\mathrm{d}s(k)}=0 \qquad \frac{\mathrm{d}f_\mu}{\mathrm{d}s(k)}=0 \qquad \frac{\mathrm{d}f^{\alpha\beta}}{\mathrm{d}s^+(k)}=0 \qquad \frac{\mathrm{d}f_\mu}{\mathrm{d}s^+(k)}=0. \tag{4.14c}$$

Equations (4.14*a*) tell us that $f^{\alpha\beta}$ and f_μ are constants; f may be absorbed in λ. Now from (4.14*b*) and (4.14*c*) we obtain

$$h_\mu(k)=\frac{k_\mu}{m}s(k)+\Phi_{(0)\mu}(k) \qquad h_\mu^+(k)=\frac{k_\mu}{m}s^+(k)+\Phi_{(0)\mu}^+(k) \tag{4.15}$$

where $\Phi_{(0)\mu}(k)$ and $\Phi_{(0)\mu}^+(k)$ are the corresponding integration constants. Having completed the integration of (4.11) and (4.12), the trajectories determined by $\mathscr{C}_\Theta$ are

$$\varepsilon^{\alpha\beta}=\varepsilon^{\alpha\beta}(\lambda^{\mu\nu}) \qquad a_\mu=\Lambda_{\nu\mu}\lambda^\nu \qquad \zeta=\zeta_0 \tag{4.16a}$$

$$\Phi_\mu(k)=[(k_\mu/m)s(k)+\Phi_{(0)\mu}(k)]\exp(-\mathrm{i}ka)$$

$$\Phi_\mu^+(k)=[(k_\mu/m)s^+(k)+\Phi_{(0)\mu}^+(k)]\exp(\mathrm{i}ka). \tag{4.16b}$$

By simply counting the integration parameters (vector fields) and the initial constants, we notice that we may dispose of one of them by shifting the parameter. In other words, a redefinition of the 'affine' parameter $s(k)$ in (4.16*b*),

$$s(k)\to s(k)-(k^\sigma/m)\Phi_{(0)\sigma}(k) \tag{4.17}$$

(and similarly for $s^+(k)$), allows us to write

$$h_\mu(k)=(k_\mu/m)s(k)+[\Phi_{(0)\mu}(k)-(k_\mu/m^2)(k^\sigma\Phi_{(0)\sigma}(k))] \tag{4.18a}$$

and, accordingly, to choose the integration constants

$$\Phi_{(0)\mu}^{\mathrm{T}}(k)=\Phi_{(0)\mu}(k)-(k_\mu/m^2)(k^\sigma\Phi_{(0)\sigma}(k)) \tag{4.18b}$$

in such a way that they are orthogonal to the 4-momentum k. With this selection the Lorentz condition

$$k^\sigma \Phi^{\mathrm{T}}_{(0)\sigma}(k)=0 \qquad k^\sigma \Phi^{\mathrm{T}+}_{(0)\sigma}(k)=0 \tag{4.19}$$

is fulfilled by the initial constants and then (4.15) becomes the decomposition of $h_\mu(k)$ and $h^+_\mu(k)$ in their longitudinal and transverse parts.

The usual decomposition of $\Phi^{\mathrm{T}}_{(0)\mu}(k)$, $\Phi^{+\mathrm{T}}_{(0)\mu}(k)$ in the transverse modes is given by

$$\Phi^{\mathrm{T}}_{(0)\mu}(k)=\sum_{i=1,2,3} C^i_{(0)}(k)\varepsilon^i_\mu(k) \qquad \Phi^{+\mathrm{T}}_{(0)\mu}(k)=\sum_{i=1,2,3} C^{+i}_{(0)}(k)\varepsilon^i_\mu(k) \tag{4.20}$$

where the 4-vectors $\varepsilon^i_\mu(k)$, $i=1, 2, 3$, fulfil the relations

$$k^\mu\varepsilon^i_\mu(k)=0 \qquad \varepsilon^i_\mu(k)\varepsilon^{j\mu}(k)=g^{ij} \qquad \sum_i \varepsilon^i_\mu(k)\varepsilon^i_\nu(k)=-\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right). \tag{4.21}$$

With this decomposition, the quantum manifold $Q=\tilde{\mathrm{G}}_{\mathrm{P}}/\mathscr{C}_\Theta$ is parametrised by

$$(C^i_{(0)}(k), C^{+i}_{(0)}(k), \zeta_0) \qquad i=1,2,3 \tag{4.22}$$

and the restrictions of Θ and $\mathrm{d}\Theta$ to Q are obtained from (4.7) and (4.8) by using (4.16), (4.20)-(4.22) with the result

$$\begin{aligned}\Theta|_{\mathrm{Q}}&=\frac{\mathrm{i}}{2}\sum_i\int_{\Omega^+_m}\mathrm{d}\Omega_k(C^{+i}_{(0)}(k)\,\mathrm{d}C^i_{(0)}(k)-C^i_{(0)}(k)\,\mathrm{d}C^{+i}_{(0)}(k))+\frac{\mathrm{d}\zeta_0}{\mathrm{i}\zeta_0}\\ \omega&=\mathrm{i}\sum_i\int_{\Omega^+_m}\mathrm{d}\Omega_k\,\mathrm{d}C^{+i}_{(0)}(k)\wedge\mathrm{d}C^i_{(0)}(k).\end{aligned} \tag{4.23}$$

Comparing with the KG case (3.15), it is seen that (4.23) is the sum of three symplectic forms written in Darboux coordinates. The symplectic structure is shown in appendix 3.

Let us now discuss briefly the Hilbert space which will again be given in terms of functionals. With the polarisation defined by

$$P=\langle\tilde{X}^{\mathrm{L}}_{\Phi_\mu(k)}, \mathscr{C}(k), \mathscr{C}^+(k), \tilde{X}^{\mathrm{L}}_{a_\mu}, \tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}\rangle \tag{4.24}$$

the equations which determine the elements $\psi\in\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{P}})$ are

$$\Xi\psi=\mathrm{i}\psi \Rightarrow \psi=\zeta\psi[\Phi_\mu(k), \Phi^+_\nu(k), \Lambda, a] \tag{4.25a}$$

$$\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}\psi=0 \Rightarrow \psi=\zeta\psi[\Phi_\mu(k), \Phi^+_\nu(k), a] \tag{4.25b}$$

$$\tilde{X}^{\mathrm{L}}_{\Phi_\alpha(k)}\psi=0 \Rightarrow \psi=\zeta\left[\exp\left(\tfrac{1}{2}\int_{\Omega^+_m}\mathrm{d}\Omega_k\,M^{\mu\nu}(k)\Phi_\mu(k)\Phi^+_\nu(k)\right)\right]\varphi[\Phi^+_\sigma(k), a] \tag{4.25c}$$

$$\tilde{X}^{\mathrm{L}}_{a_\mu}\psi=0 \Rightarrow \frac{\partial\varphi}{\partial a_\mu}+\mathrm{i}\int_{\Omega^+_m}\mathrm{d}\Omega_k\,k^\mu\Phi^+_\sigma(k)\frac{\delta\varphi}{\delta\Phi^+_\sigma(k)}=0 \tag{4.25d}$$

$$\mathscr{C}(k)\psi=0 \Rightarrow k_\sigma\frac{\delta\psi}{\delta\Phi_\sigma(k)}=0 \tag{4.25e}$$

$$\mathscr{C}^+(k)\psi=0 \Rightarrow k_\sigma\frac{\delta\psi}{\delta\Phi^+_\sigma(k)}=0. \tag{4.25f}$$

It is simple to see that (4.25*e*, *f*) imply that the functional ψ does not depend on the longitudinal part of $\Phi_\mu(k)$ and $\Phi^+_\mu(k)$. Completing the three vectors $\varepsilon^i(k)$ of (4.21) with a fourth one, $\varepsilon^0_\mu(k)=k_\mu/m$, $g^{\mu\nu}\varepsilon^\alpha_\mu(k)\varepsilon^\beta_\nu(k)=g^{\alpha\beta}$, we may write

$$\Phi_\mu(k)=\sum_{\alpha=0}^{3} C^\alpha(k)\varepsilon^\alpha_\mu(k) \qquad \Phi^+_\mu(k)=\sum_{\alpha=0}^{3} C^\alpha(k)\varepsilon^\alpha_\mu(k) \tag{4.26}$$

and check that because $k^\mu\varepsilon^\alpha_\mu(k)=g^{0\alpha}$, (4.25*e*, *f*) become

$$\frac{\delta\psi}{\delta C_0(k)}=0 \qquad \frac{\delta\psi}{\delta C^+_0(k)}=0. \tag{4.27a}$$

Using the rest of equations (4.25) we get

$$\psi=\zeta\left[\exp\left(-\tfrac{1}{2}\int_{\Omega^+_m}\mathrm{d}\Omega_k\sum_i|C^i(k)C^{+i}(k)|^2\right)\right]\varphi[C^j(k),a] \tag{4.27b}$$

$$\frac{\partial\varphi}{\partial a_\mu}+\mathrm{i}\int_{\Omega^+_m}\mathrm{d}\Omega_k\,k^\mu\sum_i C^{+i}(k)\frac{\delta\varphi}{\delta C^{+i}(k)}=0. \tag{4.27c}$$

Comparing (4.27*b*, *c*) with (3.17*c*, *d*), we see that, formally,

$$\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{P}})\approx\bigotimes_{i=1}^{3}\mathscr{H}_i(\tilde{\mathrm{G}}_{\mathrm{KG}})|_{\mathrm{diag\,Tr}_4} \tag{4.28}$$

where diag Tr_4 means that the translations (the spacetime variables) are common for all three $\mathscr{H}_i(\tilde{\mathrm{G}}_{\mathrm{KG}})$. For instance, the factorised solution of (4.27*c*) is given by

$$\varphi[C^{+i}(k),a]=\prod_{j=1}^{3}\varphi^j[C^{+j}(k),a] \tag{4.29a}$$

$$\varphi^j[C^{+j}(k),a]$$
$$=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\int_{\Omega^+_m}\mathrm{d}\Omega_k\ldots\int_{\Omega^+_m}\mathrm{d}\Omega_k\,\varphi(k_1\ldots k_n)C^{+j}(k_1)\,\mathrm{e}^{-\mathrm{i}k_1a}\ldots C^{+j}(k_n)\,\mathrm{e}^{-\mathrm{i}k_na} \tag{4.29b}$$

(cf (3.19)) and the scalar product in $\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{P}})$ is

$$(\psi',\psi)=\prod_{i=1}^{3}\int\mathrm{d}C^{+i}\,\mathrm{d}C^i\exp\left(-\int_{\Omega^+_m}\mathrm{d}\Omega_k\sum|C^l(k)|^2\right)\varphi'^*_l[C^{i+},a]\varphi_l[C^{i+},a] \tag{4.30}$$

(cf (3.20)). The discussion of the Fock space picture and the functional integration made for the KG case follows a similar pattern here.

To conclude this section we now discuss the quantum operators on $\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{P}})$. The action of the RIVF (4.5) on the functionals $\psi\in\mathscr{H}(\tilde{\mathrm{G}}_{\mathrm{P}})$ is given by

$$\tilde{X}^{\mathrm{R}}_{\Phi_\alpha(k)}\psi=\mathrm{e}^{-\mathrm{i}ka}NM^{\alpha\sigma}(k)\Phi^+_\sigma(k)\varphi \qquad \text{with}$$

$$N\equiv\zeta\exp\left(\tfrac{1}{2}\int_{\Omega^+_m}\mathrm{d}\Omega_k\,M^{\sigma\tau}(k)\Phi_\sigma(k)\Phi^+_\tau(k)\right)$$

$$\tilde{X}^{\mathrm{R}}_{\Phi^+_\alpha(k)}\psi=\mathrm{e}^{\mathrm{i}ka}N\frac{\delta\varphi}{\delta\Phi^+_\alpha(k)}$$

$$\tilde{X}^{\mathrm{R}}_{a_\mu}\psi=N\frac{\partial\varphi}{\partial a_\mu}=-\mathrm{i}N\int_{\Omega^+_m}\mathrm{d}\Omega_k\,k^\mu\Phi^+_\sigma(k)\frac{\delta\varphi}{\delta\Phi^+_\sigma(k)}=-\mathrm{i}N\int_{\Omega^+_m}\mathrm{d}\Omega_k\,k^\mu C^{+i}(k)\frac{\delta\varphi}{\delta C^{+i}(k)} \tag{4.31}$$

$$\tilde{X}^{R}_{\varepsilon^{\mu\nu}}\psi = N\delta^{\alpha\beta}_{\mu\nu} a_\alpha \frac{\delta\varphi}{\delta a_\beta} - N\int_{\Omega^+_m} d\Omega_k (m_{\mu\nu}(k)\Phi^+_\sigma(k)) \frac{\delta\varphi}{\delta\Phi^+_\sigma(k)} + N\delta^{\alpha\beta}_{\mu\nu}\int_{\Omega^+_m} d\Omega_k \Phi^+_\beta(k)\frac{\delta\varphi}{\delta\Phi^+_\alpha(k)}.$$

Thus, the action of the quantum operators on the $\varphi[C^{i+}_{(0)}(k)]$ functionals, (polarised functionals on $(\tilde{G}_P/P)/U(1)$, i.e. defining the '$C^+_{(0)}(k)$ representation') is given by

$$\begin{aligned} a^i(\boldsymbol{k}) &= \mathrm{i}\frac{\delta}{\delta C^{+i}_{(0)}(k)} \qquad a^{+i}(\boldsymbol{k}) = -\mathrm{i}C^{+i}_{(0)}(k) \\ P^\mu &= \int_{\Omega^+_m} d\Omega_k\, k^\mu \sum_i a^{+i}(\boldsymbol{k})a^i(\boldsymbol{k}) \\ M_{\mu\nu} &= \mathrm{i}\delta^{\alpha\beta}_{\mu\nu}\sum_{i,j}\int_{\Omega^+_m} d\Omega_k\, \varepsilon^j_\beta(k)\varepsilon^i_\alpha(k)a^{+i}(\boldsymbol{k})a^j(\boldsymbol{k}) \\ &\quad -\mathrm{i}\sum_{i,j}\int_{\Omega^+_m} d\Omega_k (m_{\mu\nu}(k)\varepsilon^j_\sigma(k)a^{+j}(k))\varepsilon^{i\sigma}(k)a^i(\boldsymbol{k}) \end{aligned} \tag{4.32}$$

where $C^i_{(0)}(k)$ and $C^{+i}_{(0)}(k)$ are the initial constants for the evolution given by (4.16*b*) and (4.20),

$$\begin{aligned} C^i_{(0)}(k) &= -\varepsilon^i_\sigma(k)\Phi^\sigma_{(0)}(k) \\ C^{+i}_{(0)}(k) &= -\varepsilon^i_\sigma(k)\Phi^{+\sigma}_{(0)}(k). \end{aligned} \tag{4.33}$$

5. Final comments: the Stückelberg formalism and the zero-mass limit

It is well known that, due to the gauge invariance, a description of the electromagnetic field cannot be obtained by taking the naive $m \to 0$ limit for the massive vector field. From the point of view of the Poincaré elementary systems [20] this is due to the fact that in the limit $m \to 0$ the little group for a massive particle, the rotation group, becomes the Euclidean group on the plane ($\tilde{E}_2$) and this is not a smooth transition. Because our Proca quantum group is built on the Poincaré group, it is clear that we cannot expect to obtain a description of the Maxwell field just by setting m equal to zero in (4.1), and indeed this limit is not even defined (see (4.1*b*)).

There is, however, a formalism which allows for a 'good' zero-mass limit in the sense that it leads to the right field propagators: the Stückelberg formalism (see, e.g., [16] p 136), which is amenable to our approach. In the Lagrangian formalism, the Stückelberg procedure is based on the addition of a gauge fixing term to the Lagrangian for a massive vector particle; the resulting theory describes a field whose transverse and longitudinal parts are on two different mass hyperboloids. A similar result may be achieved by the GAQ by adding to the Proca cocycle the 'gauge fixing' one

$$\xi_\lambda(g', g) = \frac{\lambda}{m^2}\int_{\Omega_{m/\sqrt{\lambda}}} d\Omega_k (\Lambda^{-1}k)^\mu k^\nu [\Phi_\mu(\Lambda'^{-1}k)\Phi^{+\prime}_\nu(k)\exp(\mathrm{i}k\Lambda' a) - \Phi^+_\mu(\Lambda'^{-1}k)\Phi'_\nu(k)\exp(-\mathrm{i}k\Lambda' a) \tag{5.1}$$

which is defined on the hyperboloid of mass $m' = m/\sqrt{\lambda}$, where m is the mass of the original Proca field and λ the dimensionless gauge fixing parameter. However, the calculations are more involved than in the Proca case and we shall omit them. Thus, the GAQ does not overcome the familiar difficulties associated with the quantisation of the Maxwell field.

As is well known, the Maxwell field presents a larger symmetry, given by the conformal group. Thus it would be more natural, perhaps, to develop the GAQ for the Maxwell field on the conformal rather than on the Poincaré group. Conformal quantum field theory [21, 22], although experimentally relevant, is not itself without difficulties (for instance, the conventional gauge fixing term violates the conformal transformation properties of the 4-potential) and we shall not discuss it here.

Appendix 1. Cohomology of G_{KG} and G_P

To find all the possible U(1) extensions of G_{KG}, the base group for the quantum group $\tilde{G}_{KG}$, it is necessary to classify the 2-$\mathbb{R}$-cocycles of G_{KG}, i.e. all the bilinear maps $\xi : G_{KG} \times G_{KG} \to \mathbb{R}$, that fulfil the conditions

$$\xi(g', g) + \xi(g'g, g'') = \xi(g, g'') + \xi(g', gg'') \tag{A1.1}$$

$$\xi(e, g) = \xi(g', e) = 0. \tag{A1.2}$$

The only vanishing Lie brackets of the generators of G_{KG} are

$$\begin{aligned} &[X^{\mathrm{L}}_{\Phi(k)}, X^{\mathrm{L}}_{\Phi^+(k)}] \\ &[X^{\mathrm{L}}_{a_\mu}, X^{\mathrm{L}}_{a_\nu}]. \end{aligned} \tag{A1.3}$$

Thus, the parameters which may be involved in an extension of G_{KG} are in principle $\Phi(k)$, $\Phi^+(k)$ and a, but due to the Lorentz transformations there is no contribution arising from the translation parameters a. The above results can also be obtained in a more rigorous way by generalising the method of Bargmann [23] to the case of infinite dimension.

Bearing in mind the formal resemblance of the group law of G_{GK} with the quantum oscillator (2.4), it can be shown that the 2-$\mathbb{R}$-cocycles are of the form (note the minus relative sign)

$$\begin{aligned} \xi(g', g) = \mathrm{i} \int & \mathrm{d}^4 k\, M(k)[\Phi'(k)\Phi^+(\Lambda'^{-1}k) \\ & \times \exp(-\mathrm{i}k\Lambda' a) - \Phi'^+(k)\Phi(\Lambda'^{-1}k) \exp(\mathrm{i}k\Lambda' a)] \end{aligned} \tag{A1.4}$$

where $M(k)$ has to be determined from the cocycle property (A1.1). A certain amount of calculation, including an integration by parts, leads to

$$M(k) = M(\Lambda k) \Rightarrow M(k) = \alpha\delta(k^2 - \beta)\theta(\pm k^0) \qquad \alpha, \beta \in \mathbb{R}. \tag{A1.5}$$

Thus, the cohomology group of G_{KG} is parametrised by the set of Lorentz-invariant distributions (A1.5), much in the same way as the mass parametrises the cohomology group of a classical free particle. Note that $M(k)$ also fixes the Poincaré orbit: only the group coordinates $\Phi(k)$ and $\Phi^+(k)$ whose index k is on the mass shell contribute to the extension of G_{KG}.

The previous reasonings for the cohomology group of G_{KG} also apply to the case of the U(1) extensions of G_P. The only non-trivial 2-ℝ-cocycles are of the form

$$\xi(g',g)=\mathrm{i}\int \mathrm{d}^4k\, M^{\mu\sigma}(\Lambda'^{-1}k)\Lambda'^{\nu}{}_{.\sigma}[\Phi^+_\mu(\Lambda'^{-1}k)\Phi'_\nu(k)\exp(-\mathrm{i}k\Lambda'a) - \Phi_\mu(\Lambda'^{-1}k)\Phi'^+_\nu(k)\exp(\mathrm{i}k\Lambda'a)] \tag{A1.6}$$

where again $M^{\mu\sigma}(k)$ is determined by (A1.1). One finds

$$M^{\mu\sigma}(k)=\alpha\delta(k^2-\beta)\theta(\pm k_0)(\lambda g^{\mu\sigma}+\gamma k^\mu k^\sigma) \qquad \alpha,\beta,\lambda,\gamma\in\mathbb{R}. \tag{A1.7}$$

Appendix 2. The Lie algebras of G_{KG} and G_P

A2.1. Calculation of the invariant vector fields

Let G be an infinite-dimensional Lie group with coordinates $g^i(k)$, characterised by $i=1,2,\ldots$, and a continuous index $k\in\Sigma$, and whose group law is written as $g''(k)=g'(k)g(k)$. The obvious generalisation of the formula

$$X_{g^i}=\left.\frac{\partial g''^j}{\partial g^i}\right|_e\frac{\partial}{\partial g^j} \tag{A2.1}$$

which gives the LIVF of an ordinary Lie group with coordinates g^i, where $i=1,2,\ldots,r$ is the finite index, is given by

$$X_{g^i(k)}=\int_\Sigma \mathrm{d}k'\left.\frac{\delta g''^j(k')}{\delta g^i(k)}\right|_e\frac{\delta}{\delta g^j(k')} \tag{A2.2}$$

where the usual partial derivatives have been substituted by functional ones. In the case of $\tilde{G}_{KG}$ and $\tilde{G}_P$, the continuous index k can be identified with the on-shell 4-momentum of a massive particle. Thus

$$\frac{\delta\Phi(k')}{\delta\Phi(k)}=\Delta_{kk'} \qquad \frac{\delta\Phi^+(k')}{\delta\Phi^+(k)}=\Delta_{kk'} \tag{A2.3}$$

for G_{KG} where $\Delta_{kk'}$ is the generalised delta function on the positive sheet of the mass hyperboloid with volume form $(2\pi)^{-3}(2E)^{-1}\,\mathrm{d}^3k$. For $\tilde{G}_P$ (A2.3) is suitably modified. Using (A2.2) and (A2.3) the expressions for the LIVF and RIVF given in (3.4), (3.5), (4.4) and (4.5) are easily obtained. For instance

$$\begin{aligned}\tilde{X}^L_{\Phi(k)}&=\int_{\Omega^+_m}\mathrm{d}\Omega_{k'}\left.\frac{\delta\Phi''(k')}{\delta\Phi(k)}\right|_{\tilde{e}}\frac{\delta}{\delta\Phi(k')}+\frac{\delta\xi(g',g)}{\delta\Phi(k)}\Xi\\ &=\int_{\Omega^+_m}\mathrm{d}\Omega_{k'}\,\Delta_{k,\Lambda'^{-1}k}\frac{\delta}{\delta\Phi(k')}-\frac{\mathrm{i}}{2}\int_{\Omega^+_m}\mathrm{d}\Omega_{k'}\,\Phi^+(k')\Delta_{k,\Lambda'^{-1}k'}\Xi\\ &=\frac{\delta}{\delta\Phi(\Lambda k)}-\tfrac{1}{2}\mathrm{i}\Phi^+(\Lambda k)\Xi\end{aligned} \tag{A2.4}$$

and, for the Lorentz vector fields,

$$\tilde{X}^R_{\varepsilon^{\mu\nu}}=\overset{R}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)\frac{\partial}{\partial\varepsilon^{\alpha\beta}}+\left.\frac{\partial}{\partial\varepsilon^{\mu\nu}}(\Lambda a)\right|_e\frac{\partial}{\partial a} +\int_{\Omega^+_m}\mathrm{d}\Omega_k\left[\left.\left(\frac{\partial\Phi(\Lambda'^{-1}k)}{\partial\varepsilon^{\mu\nu}}\right)\right|_e\frac{\delta}{\delta\Phi(k)}+\left.\left(\frac{\partial\Phi^+(\Lambda'^{-1}(k))}{\partial\varepsilon^{\mu\nu}}\right)\right|_e\frac{\delta}{\delta\Phi^+(k)}\right] \tag{A2.5}$$

where $\overset{\mathrm{R}}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)$ are functions of the parameters (see [3]). Recalling that, in our parametrisation,

$$\Lambda \in L_+^\uparrow \qquad \Lambda \equiv \exp(\tfrac{1}{2}\varepsilon^{\alpha\beta} I_{\alpha\beta}) \qquad (I_{\alpha\beta})^{\omega\tau} = \delta^{\omega\tau}_{\alpha\beta} \tag{A2.6}$$

the final expression is given by

$$\begin{aligned}\tilde{X}^{\mathrm{R}}_{\varepsilon^{\mu\nu}} = {} & \overset{\mathrm{R}}{T}{}^{\alpha\beta}_{\mu\nu}(\varepsilon)\frac{\partial}{\partial\varepsilon^{\alpha\beta}} + \delta^{\alpha\beta}_{\mu\nu} a_\beta \frac{\partial}{\partial a_\alpha} \\ & - \int_{\Omega^+_m} \mathrm{d}\Omega_k \left((m_{\mu\nu}(k)\Phi(k)) \frac{\delta}{\delta\Phi(k)} + (m_{\mu\nu}(k)\Phi^+(k)) \frac{\delta}{\delta\Phi^+(k)} \right)\end{aligned} \tag{A2.7}$$

$$m_{\mu\nu}(k) \equiv \delta^{\alpha\beta}_{\mu\nu} k_\beta \frac{\partial}{\partial k^\alpha}\bigg|_{\Omega^+_m} \tag{A2.8}$$

where $m_{\mu\nu}(k)$ is restricted to the mass-shell hyperboloid (cf (3.6)).

A2.2. Lie brackets

By means of a formal series expansion of the group composition law, it may be seen that the structure constants of an infinite-dimensional group G are (cf (A2.1) and (A2.2))

$$C^{g^i(k_1)}_{g^j(k_2), g^s(k_3)} = \frac{\delta^2 g''^i(k_1)}{\delta g'^j(k_2)\delta g^s(k_3)}\bigg|_e - \frac{\delta^2 g''^i(k_1)}{\delta g'^s(k_3)\delta g^j(k_2)}\bigg|_e \tag{A2.9}$$

where the derivatives are functional derivatives for the continuous index group coordinates and ordinary partial ones for the discrete indices. As an example, we calculate the bracket $[\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}, \tilde{X}^{\mathrm{L}}_{\Phi(k)}]$. The only non-vanishing structure constants are

$$C^{\Phi(k')}_{\varepsilon^{\mu\nu}, \Phi(k)} = \frac{\partial}{\partial\varepsilon^{\mu\nu}} \Delta_{\Lambda'^{-1}k', k} = -m_{\mu\nu}(k)\Delta_{kk'}. \tag{A2.10}$$

Thus

$$[\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}, \tilde{X}^{\mathrm{L}}_{\Phi(k)}] = \int_{\Omega^+_m} \mathrm{d}\Omega_{k'}\, C^{\Phi(k')}_{\varepsilon^{\mu\nu}, \Phi(k)} \tilde{X}^{\mathrm{L}}_{\Phi(k')} = m_{\mu\nu}(k)\tilde{X}^{\mathrm{L}}_{\Phi(k)}. \tag{A2.11}$$

The Lie brackets can also be calculated directly by using (A2.3). We do not need the explicit form of $\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}$ and $\tilde{X}^{\mathrm{R}}_{\varepsilon^{\mu\nu}}$ to find the commutators where these fields are involved: by imposing the usual Poincaré commutation relations to the Poincaré subgroups of $\tilde{G}_{\mathrm{KG}}$ and $\tilde{G}_{\mathrm{P}}$, the following equalities are obtained:

$$\begin{aligned}\overset{\mathrm{L}}{T}{}^{\alpha\beta}_{\mu\nu} \frac{\partial}{\partial\varepsilon^{\alpha\beta}}(\Lambda_{\sigma\tau}) &= -(I_{\mu\nu})_{\tau\cdot}{}^{\alpha}\Lambda_{\sigma\alpha} \\ \overset{\mathrm{R}}{T}{}^{\alpha\beta}_{\mu\nu} \frac{\partial}{\partial\varepsilon^{\alpha\beta}}(\Lambda_{\sigma\tau}) &= -(I_{\mu\nu})_{\sigma\cdot}{}^{\omega}\Lambda_{\omega\tau}.\end{aligned} \tag{A2.12}$$

For instance

$$\begin{aligned}[\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}, \tilde{X}^{\mathrm{L}}_{\Phi(k)}] &= \int_{\Omega^+_m} \mathrm{d}\Omega_{k'} (\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}} \Delta_{k'\Lambda k}) \left(\frac{\delta}{\delta\Phi(k')} - \tfrac{1}{2}\mathrm{i}\Phi^+(k')\Xi \right) \\ &= (\tilde{X}^{\mathrm{L}}_{\varepsilon^{\mu\nu}}\Lambda k)\frac{\partial}{\partial\Lambda k}\bigg|_{\Omega^+_m} \tilde{X}^{\mathrm{L}}_{\Phi(k)}.\end{aligned} \tag{A2.13}$$

We recover the former result (A2.11) if the first of (A2.12) is used. Notice the fundamental bracket which accounts for the extension of $\tilde{G}_{KG}$:

$$[\tilde{X}^{L}_{\Phi^+(k)}, \tilde{X}^{L}_{\Phi(k')}] = -i\Delta_{kk'}\Xi. \tag{A2.14}$$

The rest of the commutators of the $\tilde{G}_{KG}$ and $\tilde{G}_{P}$ algebras are calculated in the same way. The results are

Klein-Gordon	Proca
$[\tilde{X}^{L}_{\Phi^+(k)}, \tilde{X}^{L}_{\Phi(k')}] = -i\Delta_{kk'}\Xi$	$[\tilde{X}^{L}_{\Phi^+_\mu(k)}, \tilde{X}^{L}_{\Phi_\nu(k')}] = i(g^{\mu\nu} - k^\mu k^\nu/m^2)\Delta_{kk'}\Xi$
$[\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{\Phi^+(k)}] = -ik^\mu \tilde{X}^{L}_{\Phi^+(k)}$	$[\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{\Phi^+_\nu(k)}] = -ik^\mu \tilde{X}^{L}_{\Phi^+_\nu(k)}$
$[\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{\Phi(k)}] = ik^\mu \tilde{X}^{L}_{\Phi(k)}$	$[\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{\Phi_\nu(k)}] = ik^\mu \tilde{X}^{L}_{\Phi_\nu(k)}$
$[\tilde{X}^{L}_{\varepsilon^{\mu\nu}}, \tilde{X}^{L}_{\Phi(k)}] = m_{\mu\nu}(k)\tilde{X}^{L}_{\Phi(k)}$	$[\tilde{X}^{L}_{\varepsilon^{\mu\nu}}, \tilde{X}^{L}_{\Phi_\sigma(k)}] = \delta^{\lambda\sigma}_{\mu\nu}\tilde{X}^{L}_{\Phi_\lambda(k)} + m_{\mu\nu}(k)\tilde{X}^{L}_{\Phi_\sigma(k)}$
$[\tilde{X}^{L}_{\varepsilon^{\mu\nu}}, \tilde{X}^{L}_{\Phi^+(k)}] = m_{\mu\nu}(k)\tilde{X}^{L}_{\Phi^+(k)}$	$[\tilde{X}^{L}_{\varepsilon^{\mu\nu}}, \tilde{X}^{L}_{\Phi^+_\sigma(k)}] = \delta^{\lambda\sigma}_{\mu\nu}\tilde{X}^{L}_{\Phi^+_\lambda(k)} + m_{\mu\nu}(k)\tilde{X}^{L}_{\Phi^+_\sigma(k)}$
$[\tilde{X}^{L}_{\Phi(k)}, \tilde{X}^{L}_{\Phi(k')}] = 0$	$[\tilde{X}^{L}_{\Phi_\mu(k)}, \tilde{X}^{L}_{\Phi_\nu(k')}] = 0$
$[\tilde{X}^{L}_{\Phi^+(k)}, \tilde{X}^{L}_{\Phi^+(k')}] = 0$	$[\tilde{X}^{L}_{\Phi^+_\mu(k)}, \tilde{X}^{L}_{\Phi^+_\nu(k')}] = 0$

$$\text{Poincaré subalgebra}\begin{cases}[\tilde{X}^{L}_{a_\mu}, \tilde{X}^{L}_{a_\nu}] = 0\\ [\tilde{X}^{L}_{\varepsilon^{\lambda\mu}}, \tilde{X}^{L}_{\varepsilon^{\nu\rho}}] = g_{\mu\nu}\tilde{X}^{L}_{\varepsilon^{\lambda\rho}} - g_{\lambda\nu}\tilde{X}^{L}_{\varepsilon^{\mu\rho}} + g_{\lambda\rho}\tilde{X}^{L}_{\varepsilon^{\mu\nu}} - g_{\mu\rho}\tilde{X}^{L}_{\varepsilon^{\lambda\nu}}\\ [\tilde{X}^{L}_{\varepsilon^{\mu\nu}}, \tilde{X}^{L}_{a_\tau}] = \delta^{\alpha\tau}_{\mu\nu}\tilde{X}^{L}_{a_\alpha}\\ [\Xi, \underset{\text{field}}{\text{any}}] = 0.\end{cases} \tag{A2.15}$$

One can also check that the Jacobi identities and the relations $[\tilde{X}^{L}, \tilde{X}^{R}] = 0$ and $[\tilde{X}^{L}, \tilde{X}^{L}] = -[\tilde{X}^{R}, \tilde{X}^{R}]$ are fulfilled.

Appendix 3. Symplectic structure and Poisson brackets

As pointed out in § 2, the pair $(\Theta|_{\mathscr{C}_\Theta}, \tilde{G}/\mathscr{C}_\Theta \equiv Q)$ is a contact manifold from which a symplectic structure $(\omega, S \equiv Q/U(1))$ can be obtained. The Poisson brackets (PB) on $\mathscr{F}_S$—the ring of complex functions with arguments on S—give a representation of the Lie algebra of $\tilde{G}$, due to the trivial cohomology of the extended (quantum) group. In the GAQ, the RIVF generate symmetries with Hamiltonian functions taking constant values along the trajectories determined by $\mathscr{C}_\Theta$. In other words, the RIVF are the generators of the Noether invariants (conserved currents). Note that, due to the fact that $L_{\tilde{X}^R}\Theta = 0$, the Noether invariants are simply the inner product of Θ and the RIVF.

A3.1. The Klein-Gordon quantum group $\tilde{G}_{KG}$

The contact form and the symplectic form are given in (3.15*a*) and (3.15*b*). The equation $i_{X_f}\omega = -df$ implies that the Hamiltonian vector field

$$X_f = i\int_{\Omega^+_m} d\Omega_k \left(\frac{\delta f}{\delta\Phi_0(k)}\frac{\delta}{\delta\Phi^+_0(k)} - \frac{\delta f}{\delta\Phi^+_0(k)}\frac{\delta}{\delta\Phi_0(k)}\right) \tag{A3.1}$$

is the generator associated with the Noether function(al) $f = f(\Phi^+_0(k), \Phi_0(k))$. Thus, the formula for the PB is, from its usual definition $\{f, g\} = \omega(X_f, X_g)$:

$$\{f, g\} = i\int_{\Omega^+_m} d\Omega_k \left(\frac{\delta f}{\delta\Phi_0(k)}\frac{\delta g}{\delta\Phi^+_0(k)} - \frac{\delta g}{\delta\Phi^+_0(k)}\frac{\delta f}{\Phi_0(k)}\right) \tag{A3.2}$$

analogous to the expression of PB in classical mechanics, as one could have predicted just by examining ω. Note that, in particular,

$$\{\Phi_0(k), \Phi_0^+(k)\} = \mathrm{i}\Delta_{kk'}. \tag{A3.3}$$

The RIVF are given in (3.5). Contracting with Θ, the following constants of motion are obtained:

$$\begin{aligned}
&\tilde{X}^{\mathrm{R}}_{\Phi(k)} \to \mathrm{i}\Phi^+(k)\exp(-\mathrm{i}ka)\\
&\tilde{X}^{\mathrm{R}}_{\Phi^+(k)} \to -\mathrm{i}\Phi(k)\exp(\mathrm{i}ka)\\
&\tilde{X}^{\mathrm{R}}_{a_\mu} \to -\int_{\Omega_m^+} \mathrm{d}\Omega_k\, k^\mu \Phi(k)\Phi^+(k)\\
&\tilde{X}^{\mathrm{R}}_{\varepsilon^{\mu\nu}} \to \delta^{\alpha\beta}_{\mu\nu} a_\beta \int_{\Omega_m^+} \mathrm{d}\Omega_k\, k_\alpha \Phi(k)\Phi^+(k)\\
&\qquad -\frac{\mathrm{i}}{2}\int_{\Omega_m^+} \mathrm{d}\Omega_k[(m_{\mu\nu}(k)\Phi(k))\Phi^+(k) - (m_{\mu\nu}(k)\Phi^+(k))\Phi(k)].
\end{aligned} \tag{A3.4}$$

The usual expression for the 'physical' Noether invariants is recovered by taking the $\mathscr{C}_\Theta$ quotient,

$$\begin{aligned}
&\mathrm{i}\Phi_0^+(k)\\
&-\mathrm{i}\Phi_0(k)\\
&-\int_{\Omega_m^+} \mathrm{d}\Omega_k\, k^\mu \Phi_0(k)\Phi_0^+(k)\\
&\frac{\mathrm{i}}{2}\int_{\Omega_m^+} \mathrm{d}\Omega_k[(m_{\mu\nu}(k)\Phi_0^+(k))\Phi_0(k) - (m_{\mu\nu}(k)\Phi_0(k))\Phi_0^+(k)].
\end{aligned} \tag{A3.5}$$

We recognise, in the invariant associated with $\tilde{X}^{\mathrm{R}}_{a^\mu}$, the 4-momentum of the Klein–Gordon field. If we perform an integration by parts in the last expression above, we obtain the relativistic angular momentum (see, e.g., [16] p 117)

$$M_{\mu\nu} = \mathrm{i}\int_{\Omega_m^+} \mathrm{d}\Omega_k\, \Phi_0^+(k)(m_{\mu\nu}(k)\Phi_0(k)). \tag{A3.6}$$

The PB of the functions in (A3.5) fulfil the (left) Lie algebra of $\tilde{G}_{\mathrm{KG}}$, as can be deduced from its usual definition or checked by direct calculation using (A3.2).

A3.2. The Proca quantum group $\tilde{G}_P$

The case of $\tilde{G}_{\mathrm{P}}$ presents a new feature: the 2-form $\mathrm{d}\Theta/\mathscr{P}$ is only presymplectic due to the constraints on the Proca fields ($\mathscr{C}_\Theta \neq \mathscr{P}$, (4.9), (4.10)). Once the $\mathscr{C}_\Theta$ quotient is taken, one obtains, with the decomposition (4.20), that the symplectic form ω is, from (4.23), the sum of three 2-forms of the type (3.15*b*). Thus, in these Darboux coordinates, the formula for the PB is

$$\{f, g\} = \mathrm{i}\int_{\Omega_m^+} \mathrm{d}\Omega_k \sum_i \left(\frac{\delta f}{\delta C^i_{(0)}(k)}\frac{\delta g}{\delta C^{+i}_{(0)}(k)} - \frac{\delta f}{\delta C^{+i}_{(0)}(k)}\frac{\delta g}{\delta C^i_{(0)}(k)}\right). \tag{A3.7}$$

If the spatial ($i=1,2,3$) components of $\Phi^{\mu}_{(0)}(k)$ and $\Phi^{+\mu}_{(0)}(k)$ are used to parametrise the manifold of solutions, and the Lorentz condition (4.19) is taken into account, the explicit form for the PB in these new coordinates changes to

$$\{f,g\}=\mathrm{i}\int_{\Omega_m^+}\mathrm{d}\Omega_k(\delta_{ij}+k_ik_j/m^2)\left(\frac{\delta f}{\delta\Phi^i_{(0)}(k)}\frac{\delta g}{\delta\Phi^{+j}_{(0)}(k)}-\frac{\delta f}{\delta\Phi^{+i}_{(0)}(k)}\frac{\delta g}{\delta\Phi^j_{(0)}(k)}\right). \tag{A3.8}$$

Using (A3.7) and (4.21), or (A3.8) and (4.19), one can verify that

$$\{\Phi^{\mu}_{(0)}(k),\Phi^{+\nu}_{(0)}(k')\}=-\mathrm{i}(g^{\mu\nu}-k^{\mu}k^{\nu}/m^2)\Delta_{kk'}. \tag{A3.9}$$

The conserved quantities are

$$\begin{aligned}
&\tilde{X}^{\mathrm{R}}_{\Phi_\mu(k)}\to-\mathrm{i}(g^{\mu\nu}-k^{\mu}k^{\nu}/m^2)\Phi^+_{\nu}(k)\exp(-\mathrm{i}ka)\\
&\tilde{X}^{\mathrm{R}}_{\Phi^+_\mu(k)}\to\mathrm{i}(g^{\mu\nu}-k^{\mu}k^{\nu}/m^2)\Phi_{\nu}(k)\exp(\mathrm{i}ka)\\
&\tilde{X}^{\mathrm{R}}_{a_\mu}\to\int_{\Omega_m^+}\mathrm{d}\Omega_k\,k^{\mu}(g^{\rho\sigma}-k^{\rho}k^{\sigma}/m^2)\Phi_{\rho}(k)\Phi^+_{\sigma}(k)\\
&\tilde{X}^{\mathrm{R}}_{\varepsilon^{\mu\nu}}\to\frac{\mathrm{i}}{2}\int_{\Omega_m^+}\mathrm{d}\Omega_k(g^{\rho\sigma}-k^{\rho}k^{\sigma}/m^2)[\Phi_{\rho}(k)(m_{\mu\nu}(k)\Phi^+_{\sigma}(k))-\Phi^+_{\rho}(k)(m_{\mu\nu}(k)\Phi_{\sigma}(k))]\\
&\qquad+\frac{\mathrm{i}}{2}\delta^{\sigma\rho}_{\mu\nu}\int_{\Omega_m^+}\mathrm{d}\Omega_k(\delta^{\lambda}_{\rho}-k_{\rho}k^{\lambda}/m^2)(\Phi_{\lambda}(k)\Phi^+_{\sigma}(k)-\Phi^+_{\lambda}(k)\Phi_{\sigma}(k))\\
&\qquad-\delta^{\alpha\beta}_{\mu\nu}a_{\beta}\int_{\Omega_m^+}\mathrm{d}\Omega_k(g^{\lambda\sigma}-k^{\lambda}k^{\sigma}/m^2)k_{\alpha}\Phi_{\lambda}(k)\Phi^+_{\sigma}(k)
\end{aligned} \tag{A3.10}$$

and, when restricted to the quantum manifold,

$$\begin{aligned}
&\mathrm{i}\Phi^{\mu}_{(0)}(k) \qquad / \qquad k^{\mu}\Phi_{(0)\mu}(k)=0\\
&-\mathrm{i}\Phi^{+\mu}_{(0)}(k) \qquad / \qquad k^{\mu}\Phi^+_{(0)\mu}(k)=0\\
&-\int_{\Omega_m^+}\mathrm{d}\Omega_k\,k^{\mu}C^i_{(0)}(k)C^{+i}_{(0)}(k)\\
&\mathrm{i}\int_{\Omega_m^+}\mathrm{d}\Omega_k\sum_i C^{+i}_{(0)}(k)(m_{\mu\nu}(k)C^i_{(0)}(k))\\
&\qquad+\mathrm{i}\int_{\Omega_m^+}\mathrm{d}\Omega_k\sum_{ij}C^j_{(0)}(k)C^{+j}_{(0)}(k)\varepsilon^i_{\sigma}(k)(m_{\mu\nu}(k)\varepsilon^{j\sigma}(k))\\
&\qquad+\mathrm{i}\delta^{\alpha\beta}_{\mu\nu}\sum_{ij}\int_{\Omega_m^+}\mathrm{d}\Omega_k\,C^i_{(0)}(k)C^{+j}_{(0)}(k)\varepsilon^i_{\alpha}(k)\varepsilon^j_{\beta}(k).
\end{aligned} \tag{A3.11}$$

The expressions in (A3.11) can be obtained either by restricting Θ and the RIVF to the quantum manifold and then performing the inner product or by substituting in (A3.10) the trajectories given by $\mathscr{C}_\Theta$. The latter method shows the ability of the GAQ strained free system and consider the constraints only in the final step of the calculations. In this way, the generally cumbersome expressions arising when constraints are present may be avoided.

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